Fokker-Planck equation with fractional coordinate derivatives

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Abstract

Using the generalized Kolmogorov-Feller equation with long-range interaction, we obtain kinetic equations with fractional derivatives with respect to coordinates. The method of successive approximations with the averaging with respect to fast variable is used. The main assumption is that the correlator of probability densities of particles to make a step has a power-law dependence. As a result, we obtain Fokker-Planck equation with fractional coordinate derivative of order $1 < \alpha < 2$.

1 Introduction

In studying of the processes with fractal time and long-term memory a generalized kinetic equation was proposed in [1]. While the equation was of the master-type, its main property was the presence of the power-type kernel for a probability density to make a step. This type of equation was compared to the Kolmogorov-Feller equation in [2]. In this paper, we would like to go farther and to show the conditions under which one can obtain the fractional generalization of the Fokker-Planck equation from the Fokker-Planck equation.

Fractional calculus [3, 4, 5] has found many applications in recent studies in mechanics and physics, and the interest in fractional equations has been growing continually during the last years. Fractional Fokker-Planck equations with coordinate and time derivatives of non-integer order has been suggested in [6]. The solutions and properties of these equations are described in Refs. [2, 7]. The Fokker-Planck equation with fractional coordinate derivatives was also considered in [8, 9, 10, 11, 12].

The Kolmogorov-Feller equation is integro-differential one and it belongs to the type of master equations broadly used in different physical applications. It is well-known that the Kolmogorov-Feller equation can lead us to the Fokker-Planck equation [13] under some conditions. In this paper, we use the method of successive approximations with the averaging with respect to the fast variable [14]. The main assumption is that the correlator of probability densities of particle to make a step has power-law dependence. As a result, we obtain Fokker-Planck equations with fractional derivatives of order $1 < \alpha < 2$.

In Sec. 2, the Kolmogorov-Feller equation for one-dimensional case is considered. In Sec. 3, we present a generalization of the Kolmogorov-Feller equation for two-dimensional case. The method of successive approximations is used for this generalized equation in Sec. 4. In Sec. 5, we use averaging with respect to the fast variable to derive fractional Fokker-Planck equations. Finally, a short conclusion is given in Sec. 6.

2 Kolmogorov-Feller equation for one-dimensional case

2.1 Operator representation of the KF-equation

Let $P(t,x)$ be a probability density to find a particle at x at time instant t. The normalization condition for $P(t,x)$ is

$$\int_{-\infty}^{+\infty} dx\, P(t,x) = 1 \quad (t > 0).$$

The Kolmogorov-Feller (KF) equation has the form

$$\frac{\partial P(t,x)}{\partial t} = \int_{-\infty}^{+\infty} dx'\, w(x')[P(t,x-x') - P(t,x)], \quad P(0,x) = \delta(x), \tag{1}$$

where $w(x')$ is probability density of particle to make a step of the length x', and

$$\int_{-\infty}^{+\infty} dx'\, w(x') = 1. \tag{2}$$

Let us introduce the operator representation of the KF-equation. We define the translation operator

$$T_{x'} = \exp\{-x'\partial_x\}, \tag{3}$$

such that

$$T_{x'} P(t,x) = P(t,x-x'), \tag{4}$$

and the finite difference operator

$$\Delta_{x'} = I - T_{x'}, \tag{5}$$

where I is an identity operator. Then the Kolmogorov-Feller equation (1) can be presented as

$$\frac{\partial P(t,x)}{\partial t} = L(\Delta)\, P(t,x). \tag{6}$$

Here we use the integro-differential operator

$$L(\Delta) = -\int_{-\infty}^{+\infty} dx'\, w(x')\, \Delta_{x'}. \tag{7}$$

The operator (7) will be called the Kolmogorov-Feller operator.

2.2 KF-equation with fractional coordinate derivative

The well-known fractional Caputo derivative [4] of order α is defined by

$$ {}^{C}D_x^{\alpha} P(x) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{x} \frac{dz}{(x-z)^{\alpha}} \frac{\partial P(z)}{\partial z}, \quad (0 < \alpha < 1). \tag{8} $$

The fractional Marchaud derivative [4] of order α is defined by

$$ \mathbf{D}_x^{\alpha} P(x) = \frac{1}{\Gamma(-\alpha)} \int_{-\infty}^{x} [P(z) - P(x)] \frac{dz}{(x-z)^{\alpha+1}}, \quad (0 < \alpha < 1). \tag{9} $$

Using $x' = x - z$, equation (9) has the form

$$ \mathbf{D}_x^{\alpha} P(x) = \frac{1}{\Gamma(-\alpha)} \int_{0}^{\infty} \frac{dx'}{(x')^{\alpha+1}} [P(x-x') - P(x)]. $$

If the function $w(x')$ in KF-equation (1) is the exponential function

$$ w(x') = \frac{a}{x'^{\alpha+1}} H(x'), \tag{10} $$

where $H(x')$ is a Heaviside step function, then Eq. (1) can be presented through the fractional coordinate derivative

$$ \frac{\partial P(t,x)}{\partial t} = a \mathbf{D}_x^{\alpha} P(t,x), \quad P(0,x) = \delta(x), \quad (0 < \alpha < 1). \tag{11} $$

This is the Kolmogorov-Feller equation with fractional coordinate derivatives of order $0 < \alpha < 1$. Note that the function $w(x)$ is a probability density and it should satisfy the normalization condition (2). Then equation (10) can be considered only as an approximation for large x'.

2.3 Generalized KF-equation

In the general case, we can suppose that probability density of particle to make a step of the length x' depends on the time instant t and the coordinate x. Then we should replace $w(x')$ by $w(t, x|x')$ in KF-equation (1). As a result, we can consider the equation

$$ \frac{\partial P(t,x)}{\partial t} = \varepsilon \int_{-\infty}^{+\infty} dx' \, w(t, x|x') [P(t, x-x') - P(t,x)], \quad P(0,x) = \delta(x), \tag{12} $$

where ε is a small parameter. Here $w(t, x|x')$ is the probability density to make a step of the length x' at the time instant t from the coordinate x. If $w(t, x|x') = w(x')$, then Eq. (12) gives Eq. (1). We can assume that during any interval of time $(t, t + dt)$ the value of the variable $x(t)$ remains equal to x with probability $1 - p(t, x)dt$ and may undergo a change only with probability $p(t, x)dt$ (see more about this equation in Sec.55. of [13]). Then

$$w(t, x|x') = p(t, x)w(x'), \tag{13}$$

where $p(t, x)$ is a bounded function. If $p(t, x) = 1$, then equations (12) and (13) gives Eq. (1).

Using the operator (5), equation (12) can be presented in the operator form

$$\frac{\partial P(t, x)}{\partial t} = \varepsilon L(t, x, \Delta)\, P(t, x), \tag{14}$$

where

$$L(t, x, \Delta) = -\int_{-\infty}^{+\infty} dx'\, w(t, x|x')\, \Delta_{x'}. \tag{15}$$

Equation (14) will be called a generalized KF-equation for one-dimensional case.

2.4 Successive approximations

The generalized Kolmogorov-Feller equation (14) can be rewritten in the integral form

$$P(t, x) - P(0, x) = \int_0^t d\tau\ L(t, x, \Delta)P(\tau, x). \tag{16}$$

This equation can be presented as the integral Volterra type equation

$$P(t, x) = P(0, x) + \varepsilon \int_0^t dt_1\, L(t_1, x, \Delta)P(t_1, x). \tag{17}$$

Let us consider the successive approximations. Substitution of equation (17) in the form

$$P(t_1, x) = P(0, x) + \varepsilon \int_0^{t_1} dt_2\, L(t_2, x\Delta)P(t_2, x) \tag{18}$$

into equation (17) gives

$$P(t,x) = P(t_0,x) + \varepsilon \int_0^t dt_1\, L(t_1,x,\Delta)P(0,x)+$$

$$+\,\varepsilon^2 \int_0^t dt_1 \int_0^{t_1} dt_2\, L(t_1,x,\Delta)L(t_2,x,\Delta)P(t_2,x). \tag{19}$$

Changing the variables $t_1 \to t_2$, and $t \to t_1$ in equation (18), and substituting into (19), we obtain an equation up to ε^2 in the form

$$P(t,x) = P(0,x) + \varepsilon \int_0^t dt_1\, L(t_1,x,\Delta)P(0,x)+$$

$$+\,\varepsilon^2 \int_0^t dt_1 \int_0^{t_1} dt_2\, L(t_1,x,\Delta)L(t_2,x,\Delta)P(0,x). \tag{20}$$

If the function $w(x')$ is the exponential function (10), then $L(t,x,\Delta)$ is a differential operator of order $0 < \alpha < 1$ with respect to x, and $L(t_1,x,\Delta)L(t_2,x,\Delta)P(t_2,x)$ is a differential operator of the order $0 < 2\alpha < 2$. To obtain fractional kinetic equations of the order $0 < 2\alpha < 2$, we should consider the average procedure before a partial differentiation of Eq. (20) with respect to time is realized. Without averaging, we derive equation of order $0 < \alpha < 1$.

3 Distribution function and Kolmogorov-Feller equation for two-dimensional case

Let $P(t,x,y)$ be a function of probability distribution in a phase space. The variables x and y describe the phase space of a system. There are the following interpretations for the variables x and y.

- A system with one degree of freedom can be presented by momentum $x = p$, and coordinates $y = q$.

- A system can be described by action $x = I$, and phase $y = \theta$.

- n-particle system can be defined by $x = (q_1, p_1)$ and $y = (q_2, p_2, ..., q_n, p_n)$, or $x = q_1$, and $y = (p_1, q_2, p_2, ..., q_n, p_n)$.
- The variables x describe a system, and y describes an environment of this system.

We plan to use the reduced distribution and average values with respect to y, where y will be considered as a fast variable.

3.1 Generalized KF-equation for two-dimensional case

We assume that $x \in X \subset \mathbb{R}$ and $y \in Y \subset \mathbb{R}$, then $\mathbf{r} = (x, y) \in X \times Y \subset \mathbb{R}^2$. We plan to consider x is a slow variable and y will be considered as fast variable. The distribution function $P(t, \mathbf{r})$ in the region $X \times Y$ satisfies the generalized Kolmogorov-Feller equation

$$\frac{\partial P(t, \mathbf{r})}{\partial t} = \varepsilon \int_{X \times Y} d^2 r_1 \, w(t, \mathbf{r}|\mathbf{r}_1)[P(t, \mathbf{r} - \mathbf{r}_1) - P(t, \mathbf{r})], \tag{21}$$

where $d^2 r_1 = dx_1 dy_1$, and ε is a small parameter. Here $w(t, \mathbf{r}|\mathbf{r}_1)$ is the probability density to make a step on the vector $\mathbf{r}_1$ at the time instant t from the point $\mathbf{r}$.

Equation (21) can be presented in the form

$$\frac{\partial}{\partial t} P(t, \mathbf{r}) = \varepsilon L(t, \mathbf{r}, \Delta) P(t, \mathbf{r}), \tag{22}$$

where $L(t, \mathbf{r}, \Delta)$ is a Kolmogorov-Feller operator

$$L(t, \mathbf{r}, \Delta) = -\int_{X \times Y} d^2 r_1 \, w(t, \mathbf{r}|\mathbf{r}_1) \, \Delta_{\mathbf{r}_1}, \tag{23}$$

Here $\Delta_{\mathbf{r}_1}$ is a finite difference operator

$$\Delta_{\mathbf{r}_1} = I - T_{\mathbf{r}_1},$$

where $T_{\mathbf{r}_1}$ is a translation operator in $X \times Y$ that is defined by

$$T_{\mathbf{r}_1} = \exp\{-\mathbf{r}_1 \nabla\} = \exp\{-x_1 \partial_x - y_1 \partial_y\}.$$

We can assume that during any interval of time $(t, t+dt)$ the value of the variable $\mathbf{r}(t)$ remains equal to $\mathbf{r}$ with probability $1 - p(t, \mathbf{r})dt$ and may undergo a change only with probability $p(t, \mathbf{r})dt$ (see Sec.55. in [13]). Then

$$w(t, \mathbf{r}|\mathbf{r}') = p(t, \mathbf{r})\, w(\mathbf{r}'), \tag{24}$$

where $p(t, \mathbf{r})$ is a bounded function.

We can use the variables x, y instead of $\mathbf{r}$. Then equation (21) for the distribution function $P(t, x, y)$ will be presented in the form

$$\frac{\partial P(t,x,y)}{\partial t} = \varepsilon \int_{X \times Y} dx_1 dy_1\, w(t, x, y|x_1, y_1)[P(t, x - x_1, y - y_1) - P(t, x, y)]. \tag{25}$$

This equation can be rewritten as

$$\frac{\partial}{\partial t} P(t, x, y) = \varepsilon L(t, x, y, \Delta_x, \Delta_y) P(t, x, y), \tag{26}$$

where $L(t, x, y, \Delta_x, \Delta_y)$ is a Kolmogorov-Feller operator

$$L(t, x, y, \Delta_x, \Delta_y) = \int_{X \times Y} dx_1 dy_1\, w(t, x, y|x', y')\, [T_{x_1} T_{y_1} - I]. \tag{27}$$

This is the generalized Kolmogorov-Feller equation in the operator form.

4 Method of successive approximations

The generalized Kolmogorov-Feller equation

$$\frac{\partial}{\partial t} P(t, \mathbf{r}) = \varepsilon L(t, \mathbf{r}, \Delta) P(t, \mathbf{r}) \tag{28}$$

can be presented as the integral Volterra equation

$$P(t, \mathbf{r}) = P_0(\mathbf{r}) + \varepsilon \int_0^t dt_1\, L(t_1, \mathbf{r}, \Delta) P(t_1, \mathbf{r}), \tag{29}$$

where $P_0(\mathbf{r}) = P(0, \mathbf{r})$. Substitution of equation (29) in the form

$$P(t_1, \mathbf{r}) = P_0(\mathbf{r}) + \varepsilon \int_0^{t_1} dt_2\, L(t_2, \mathbf{r}, \Delta) P(t_2, \mathbf{r}) \tag{30}$$

into Eq. (29) gives

$$P(t,\mathbf{r}) = P_0(\mathbf{r}) + \varepsilon \int_0^t dt_1\, L(t_1,\mathbf{r},\Delta)P_0(\mathbf{r}) +$$

$$+\,\varepsilon^2 \int_0^t dt_1 \int_0^{t_1} dt_2\, L(t_1,\mathbf{r},\Delta)L(t_2,\mathbf{r},\Delta)P(t_2,\mathbf{r}). \tag{31}$$

Changing the variables $t_1 \to t_2$, and $t \to t_1$ in equation (30), and substituting into (31), we get

$$P(t,\mathbf{r}) = P_0(\mathbf{r}) + \varepsilon \int_0^t dt_1\, L(t_1,\mathbf{r},\Delta)P_0(\mathbf{r}) +$$

$$+\,\varepsilon^2 \int_0^t dt_1 \int_0^{t_1} dt_2\, L(t_1,\mathbf{r},\Delta)L(t_2,\mathbf{r},\Delta)P_0(\mathbf{r}) + ... \tag{32}$$

Using the chronological multiplication

$$T\{L(t_1,\mathbf{r},\Delta)L(t_2,\mathbf{r},\Delta)\} = \begin{cases} L(t_1,\mathbf{r},\Delta)L(t_2,\mathbf{r},\Delta) & t_1 > t_2; \\ L(t_2,\mathbf{r},\Delta)L(t_1,\mathbf{r},\Delta) & t_2 > t_1, \end{cases} \tag{33}$$

equation (32) can be symmetric with respect to t_1 and t_2:

$$P(t,\mathbf{r}) = P_0(\mathbf{r}) + \varepsilon \int_0^t dt_1\, L(t_1,\mathbf{r},\Delta)P_0(\mathbf{r}) +$$

$$+\,\frac{1}{2}\varepsilon^2 \int_0^t dt_1 \int_0^{t} dt_2\, T\{L(t_1,\mathbf{r},\Delta)L(t_2,\mathbf{r},\Delta)\}P_0(\mathbf{r}) + ... \tag{34}$$

This is the symmetric representation of equation (32).

5 Averaging with respect to the fast variable

In this section, let us consider the variables $\mathbf{r} = (x, y)$ as slow (x) and fast (y). Substitution of (23) into (34) gives

$$P(t,\mathbf{r}) = P_0(\mathbf{r}) - \varepsilon \int_0^t dt_1 \int_{X\times Y} d^2r_1\, w(t_1,\mathbf{r}|\mathbf{r}_1)\, \Delta_{\mathbf{r}_1} P_0(\mathbf{r}) +$$

$$+\,\frac{1}{2}\varepsilon^2 \int_0^t dt_1 \int_0^t dt_2 \int_{X\times Y} d^2r_1 \int_{X\times Y} d^2r_2\, T\{w(t_1,\mathbf{r}|\mathbf{r}_1)\, \Delta_{\mathbf{r}_1}\, w(t_2,\mathbf{r}|\mathbf{r}_2)\, \Delta_{\mathbf{r}_2}\} P_0(\mathbf{r}) + O(\varepsilon^3). \tag{35}$$

The function $w(t, \mathbf{r}|\mathbf{r}_1)$ is the probability density to make a step of the vector $\mathbf{r}_1$ at the time instant t from the point $\mathbf{r}$. The first assumption is that this probability density has a weak dependence (up to terms of order $O(\varepsilon)$) on the point $\mathbf{r}$, i.e.,

$$w(t, \mathbf{r} + \mathbf{r}_1|\mathbf{r}_2) = w(t, \mathbf{r}|\mathbf{r}_2) + O(\varepsilon). \tag{36}$$

Then

$$\Delta_{\mathbf{r}_1} w(t_2, \mathbf{r}|\mathbf{r}_2) = 0,$$

and

$$w(t_1, \mathbf{r}|\mathbf{r}_1)\, \Delta_{\mathbf{r}_1} w(t_2, \mathbf{r}|\mathbf{r}_2)\, \Delta_{\mathbf{r}_2} = w(t_1, \mathbf{r}|\mathbf{r}_1)\, w(t_2, \mathbf{r}|\mathbf{r}_2)\, \Delta_{\mathbf{r}_1} \Delta_{\mathbf{r}_2}.$$

As a result, equation (35) has the form

$$P(t, \mathbf{r}) = P_0(\mathbf{r}) - \varepsilon \int_0^t dt_1 \int_{X\times Y} d^2r_1\, w(t_1, \mathbf{r}|\mathbf{r}_1)\, \Delta_{\mathbf{r}_1} P_0(\mathbf{r}) +$$

$$+\frac{1}{2}\varepsilon^2 \int_0^t dt_1 \int_0^t dt_2 \int_{X\times Y} d^2r_1 \int_{X\times Y} d^2r_2\, T\{w(t_1, \mathbf{r}|\mathbf{r}_1)\, w(t_2, \mathbf{r}|\mathbf{r}_2)\}\, \Delta_{\mathbf{r}_1} \Delta_{\mathbf{r}_2} P_0(\mathbf{r}) + O(\varepsilon^3).$$

The second assumption states that $P_0(\mathbf{r}) = P_0(x, y)$ does not depend on the fast variable y up to ε-terms such that

$$P_0(\mathbf{r}) = P_0(x, y) = \rho_0(x) + O(\varepsilon). \tag{37}$$

Then, we have

$$\Delta_{\mathbf{r}_1} \Delta_{\mathbf{r}_2} P_0(\mathbf{r}) = \Delta_{x_1} \Delta_{x_2} \rho_0(x) + O(\varepsilon). \tag{38}$$

Averaging over the variable y will be denoted by $< >_y$. We also use the notations

$$\rho(t, x) =< P(t, \mathbf{r}) >_y, \tag{39}$$

and $\rho(0, x) = \rho_0(x)$.

Averaging of equation (35) with respect to the fast variable y, we obtain

$$\rho(t, x) = \rho_0(x) - \varepsilon \int_0^t dt_1 \int_X dx_1\, A(t_1, x|x_1)\, \Delta_{x_1} \rho_0(x) +$$

$$+\frac{1}{2}\varepsilon^2 \int_0^t dt_1 \int_X dx_1 \int_X dx_2\, B(t_1,x|x_1,x_2)\, \Delta_{x_1}\, \Delta_{x_2}\, \rho_0(x) + O(\varepsilon^3), \qquad (40)$$

where we introduce the functions

$$A(t_1,x|x_1) = \int_Y dy_1\ < w(t_1,\mathbf{r}|\mathbf{r}_1) >_y, \qquad (41)$$

$$B(t_1,x|x_1,x_2) = \int_0^t dt_2 \int_Y dy_1 \int_Y dy_2\ < T\{w(t_1,\mathbf{r}|\mathbf{r}_1)\, w(t_2,\mathbf{r}|\mathbf{r}_2)\} >_y . \qquad (42)$$

Using $\mathbf{r} = (x,y)$, these functions can be presented by

$$A(t_1,x|x_1) = \int_Y dy_1\ < w(t_1,x,y|x_1,y_1) >_y, \qquad (43)$$

$$B(t_1,x|x_1,x_2) = \int_0^t dt_2 \int_Y dy_1 \int_Y dy_2\ < T\{w(t_1,x,y|x_1,y_1)\, w(t_2,x,y|x_2,y_2)\} >_y . \qquad (44)$$

The third assumption is that the function $B(t_1,x|x_1,x_2)$ is diagonal with respect to variables x_1 and x_2 up to ε-term, i.e.,

$$B(t_1,x|x_1,x_2) = B(t_1,x|x_1)\delta(x_1-x_2) + O(\varepsilon). \qquad (45)$$

Then the operator $\Delta_{x_1}\,\Delta_{x_2}$ will be the finite difference operator $\Delta^2_{x_1}$ of second order. This allows us to have fractional derivative for the exponential function $B(t,x|x_1)$ since Marchaud and Riesz fractional derivatives [3] are defined through the finite difference operator.

The fourth assumption is that the functions $A(t_1,x|x_1)$ and $B(t_1,x|x_1)$ are exponential functions up to ε such that

$$A(t,x|x_1) = a(t)\frac{1}{\kappa(\alpha_1,1)}\frac{1}{|x_1|^{\alpha_1+1}}H(x_1) + O(\varepsilon), \quad (0<\alpha_1<1), \qquad (46)$$

$$B(t,x|x_1) = b(t,x)\frac{1}{\kappa(\alpha_2,2)}\frac{1}{|x_1|^{\alpha_2}}H(x_1) + O(\varepsilon), \quad (1<\alpha_2<2), \qquad (47)$$

where $H(x)$ is the Heaviside step function, and

$$\kappa(\alpha,n) = -\Gamma(\alpha)A_n(\alpha), \quad A_n(\alpha) = \sum_{k=0}^{n}(-1)^{k-1}\frac{n!}{k!(n-k)!}k^{\alpha}, \qquad (48)$$

with $n-1<\alpha<n$.

As a result, Eq. (40) gives

$$\rho(t,x)=\rho_0(x)+\varepsilon\int_0^t dt_1\, a(t_1)\mathbf{D}_x^{\alpha_1}\rho_0(x)+\varepsilon^2\int_0^t dt_1\, b(t_1,x)\mathbf{D}_x^{\alpha_2}\rho_0(x)+O(\varepsilon^3). \tag{49}$$

Here $\mathbf{D}_x^{\alpha}$ is Marchaud fractional derivative [3] of order α with respect to x, which is defined by the equation

$$\mathbf{D}_x^{\alpha}f(x)=\frac{1}{\kappa(\alpha,n)}\int_0^{\infty}\frac{\Delta_y^n f(x)}{y^{\alpha+2-n}}dy, \quad (n-1<\alpha<n). \tag{50}$$

where Δ_y^n is a finite difference of order n such that

$$\Delta_y^n f(x)=(I-T_y)^n f(x)=\sum_{m=0}^{n}(-1)^m\frac{n!}{m!(n-m)!}f(x-my). \tag{51}$$

In general, the variables x and x_1 can be vectors in the N-dimensional space $\mathbb{R}^N$, where $N=1,2,3,....$ The fourth assumption for the functions $A(t_1,x|x_1)$ and $B(t_1,x|x_1)$ can be realized in the form other than (46) and (47). We can suppose that the functions $A(t_1,x|x_1)$ and $B(t_1,x|x_1)$ are exponential functions up to ε such that

$$A(t,x|x_1)=a(t)\frac{1}{d_N(1,\alpha_1)}\frac{1}{|x_1|^{N+\alpha_1}}+O(\varepsilon), \quad (0<\alpha_1<1), \tag{52}$$

$$B(t,x|x_1)=b(t,x)\frac{1}{d_N(2,\alpha_2)}\frac{1}{|x_1|^{N+\alpha_2}}+O(\varepsilon), \quad (1<\alpha_2<2), \tag{53}$$

where $x,x_1\in\mathbb{R}^N$, and

$$d_N(n,\alpha)=\frac{2^{-\alpha}\pi^{1+N/2}A_n(\alpha)}{\sin(\alpha\pi/2)\Gamma(1+\alpha/2)\Gamma((N+\alpha)/2)} \tag{54}$$

with $n-1<\alpha<n$. As a result, Eq. (40) gives fractional equation of the form (49), where $\mathbf{D}_x^{\alpha}$ is the fractional Riesz derivative (see [3] Sec.25.4) of order α with respect to $x\in\mathbb{R}^N$, which is defined by the equation

$$\mathbf{D}_x^{\alpha}f(x)=\frac{1}{d_N(n,\alpha)}\int_{\mathbb{R}^N}\frac{\Delta_y^n f(x)}{|y|^{\alpha+N}}\,d^Ny, \quad (n-1<\alpha<n). \tag{55}$$

To denote Riesz fractional derivative (55) is also used the notation $(-\Delta)^{\alpha/2}$. Note that the Fourier transform $\mathcal{F}$ of this derivative (see Property 2.34 of [4]) is defined by

$$\Big(\mathcal{F}\{\mathbf{D}^{\alpha}_{x} f(x)\}\Big)(\mathbf{k}) = |\mathbf{k}|^{\alpha} \Big(\{\mathcal{F} f(x)\}\Big)(\mathbf{k}).$$

The representation of the assumption in the form (52) and (53) allows us to obtain fractional kinetic equation for arbitrary N-dimensional space (for example, in the 3-dimensional Euclidean space).

The partial time differentiation of equation (49) gives

$$\frac{\partial}{\partial t}\rho(t,x) = \varepsilon a(t)\mathbf{D}^{\alpha_1}_{x}\rho_0(x) + \varepsilon^2 b(t,x)\mathbf{D}^{\alpha_2}_{x}\rho_0(x) + O(\varepsilon^3). \tag{56}$$

Substitution of equation (49) in the form

$$\rho_0(x) = \rho(t,x) - \varepsilon \int_0^t dt_1\, a(t_1)\mathbf{D}^{\alpha_1}_{x}\rho_0(x) + O(\varepsilon^2) \tag{57}$$

into equation (56) gives

$$\frac{\partial}{\partial t}\rho(t,x) = \varepsilon a(t)\mathbf{D}^{\alpha_1}_{x}\rho(t,x) + \varepsilon^2 \Big(b(t,x)\mathbf{D}^{\alpha_2}_{x} - c(t)\mathbf{D}^{2\alpha_1}_{x}\Big)\rho(t,x) + O(\varepsilon^3), \tag{58}$$

where

$$c(t) = a(t)\int_0^t dt_1\, a(t_1).$$

Equation (58) up to $O(\varepsilon^3)$ has the form

$$\frac{\partial}{\partial t}\rho(t,x) = \varepsilon a(t)\mathbf{D}^{\alpha_1}_{x}\rho(t,x) + \varepsilon^2 \Big(b(t,x)\mathbf{D}^{\alpha_2}_{x} - c(t)\mathbf{D}^{2\alpha_1}_{x}\Big)\rho(t,x). \tag{59}$$

This is the fractional kinetic equation with noninteger derivatives of the order $1 < \alpha_2 < 2$ and $0 < 2\alpha_1 < 2$ with respect to coordinate x.

If $a(t) = 0$, i.e.. $A(t,x|x_1) = 0$, then we have the fractional equation of order $1 < \alpha_2 < 2$ with respect to x such that

$$\frac{\partial}{\partial t}\rho(t,x) = \varepsilon^2 b(t,x)\mathbf{D}^{\alpha_2}_{x}\rho(t,x). \tag{60}$$

This is the fractional Fokker-Planck equation that is suggested in [6] to describe fractional kinetics. The solutions and properties of these equations are described in Refs. [2, 7]. The Fokker-Planck equation with fractional coordinate derivatives are also considered in [8, 9, 10, 11, 12].

6 Conclusion

In this paper, the Fokker-Planck equations with coordinate derivatives of non-integer order $1 < \alpha < 2$ are derived by using Kolmogorov-Feller equation. The method of successive approximations and the averaging with respect to a fast variable are used. The main assumption in the paper is that the correlator of two probability densities of particle to make a step has the form of exponential function. As a result, the Fokker-Planck equations can have derivatives of order $1 < \alpha < 2$.

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